gauge and anomalous gauge theories with massive fermions

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(October 13, 2000)

Abstract

Using a synthesis of the functional integral and operator approaches we discuss the fermion-boson mapping and the role played by the Bose field algebra in the Hilbert space of two-dimensional gauge and anomalous gauge field theories with massive fermions. In the QED_2 with quartic self-interaction among massive fermions, the use of an auxiliary vector field introduces a redundant Bose field algebra that should not be considered as an element of the intrinsic algebraic structure defining the model. In the anomalous chiral QED_2 with massive fermions the effect of the chiral anomaly leads to the appearance in the mass operator of a spurious Bose field combination. This phase factor carries no fermion selection rule and the expected absence of θ-vacuum in the anomalous model is displayed from the operator solution. Even in the anomalous model with massive Fermi fields, the introduction of the Wess-Zumino field replicates the theory, changing neither its algebraic content nor its physical content.

I. INTRODUCTION

In the recent efforts towards the extension of the functional integral bosonization to $2+1$ dimensions [1,2], use has been made of an interpolating field procedure that leads to a "mapping" of the partition function of the original theory into a partition function of Chern-Simons-type theories. Unfortunately, until recent investigations of the fermion-boson mappings in $2+1$ dimensions were limited to perturbative analysis and these mappings generally are established on the level of factorizable partition functions and not established on the Hilbert space of states.

The Thirring model in $2+1$ dimensions has been considered in Refs. [1,2]. In the Abelian case, it has been shown that to lowest order in inverse fermion mass, the Thirring model partition function coincides with that of the Maxwell-Chern-Simons theory. In the non-Abelian case, and in the low-energy regime, it has been shown that the bosonized partition function of the $SU(N)$ massive Thirring model is related to $SU(N)$ Yang-Mills-Chern-Simons gauge theory. In both cases, the bosonization is performed using an auxiliary vector field and the correspondence between the models are established on the level of the partition functions.

At the present state of the research, and due to a large number of papers on the subject, it seems to be very instructive to make a investigation of the basic structural aspects involved in the functional integral bosonization approach using the auxiliary vector field. The bosonization of two-dimensional gauge and anomalous gauge models has been reexamined quite recently in Refs. [3–6]. In Ref. [3] using the functional integral formulation we reconstruct in the Hilbert space of states the Coleman's proof [8] of the fermion-boson mapping between the massive Thirring and sine-Gordon theories. We show that the use of an auxiliary vector field enlarges the Hilbert space by the introduction of an external Bose field algebra that should not be considered as an element of the intrinsic algebraic structure defining the model. The factorization of the partition function generally lead to incorrect conclusion concerning the physical content of the model [3,5,6].

The general procedure to establish the fermion-boson mappings is performed by using one of the quantum field theory approaches, the operator approach or the functional integral approach. However, many structural aspects of the models are not easily accessible or not visible at all in one of these approaches. In our presentation, which interpolates between the functional integral and operator approaches, close attention is paid to mantaining complete control on the Hilbert space structure needed for the representation of the intrinsic field algebra, whose Wightman functions define the model. The subject of this paper is to analyze non-perturbative aspects and the Hilbert space structure of two-dimensional gauge and anomalous gauge models with a procedure which interpolates between the operator and functional integral formalisms. In this way, the conclusions about the structural aspects of the models emerges as a synthesis of the analysis performed using the two quantum field theory approaches.

In order to obtain insight into the role played by the auxiliary vector field in the Hilbert space of the bosonized theory, in the first part of the present work we shall consider the functional integral bosonization of the massive QED_2 with quartic fermion self-interaction (massive Schwinger-Thirring model). Using the functional integral bosonization we obtain the operator solution and the fermion-boson mapping is established in the Hilbert space of states. We show that the use of the auxiliary vector field to reduce the Thirring interaction to a quadratic form, introduces a redundant Bose field algebra that should not be considered as an element of the intrinsic algebraic structure defining the model. The infinitely delocalized states generated from the Bose field combination with zero scale dimension do not belong to the Hilbert space of the model. This streamlines the discussion present in Refs. [3,9,11].

In the second part of the paper we consider the functional integral bosonization of the anomalous chiral QED_2 with massive fermions. In Refs. [5,6] some structural aspects of the bosonization of anomalous two-dimensional models with massless fermions has been considered. The effective bosonized partition function of the massless chiral QED_2 (massless chiral Schwinger model) factorizes into the partition functions of a free massive vector field, a massless free field carrying the free fermion selection rules and two "decoupled" massless free fields quantized with opposite metric. In the model with massless fermions, the extraction of these "decoupled" free and massless Bose fields from the operator solution [10], by performing an improper factorization of the Hilbert space representing the intrinsic field algebra, leads to some misleading conclusions about basic structural properties of the model, such as the equivalence of the Schwinger model with the chiral model defined for the regularization depending parameter $a = 2$, the violation of the asymptotic factorization property of the Wightman functions and the existence of θ-vacuum. These features cannot be regarded as being structural properties of the anomalous model since they are dependent of the use of a redundant Bose field algebra, rather than on the field algebra which defines the model. The BRST constraints and the Hilbert space structure of the isomorphic gauge noninvariant and gauge invariant bosonized formulations of chiral QCD_2 with massless fermions was analyzed in Ref. [6]. The BRST subsidiary conditions are found not to provide a sufficient criterium for defining physical states in the Hilbert space and additional superselection rules must be taken into account.

In order to probe the effect of the anomaly in the mass term for the Fermi field, we consider the anomalous chiral QED_2 with massive fermions. We show that the "decoupled" fields quantized with opposite metric acquire a non-trivial dynamics and are coupled by a sine-Gordon-like interaction. Nevertheless, their combination is a massless free field which generates zero norm states from the vacuum. The effect of the chiral anomaly leads to the appearance of a spurious Bose field combination in the mass operator. This phase factor signalize the explicit breakdown of the chiral symmetry and its only effect in the operator solution is to ensure the invariance of the field algebra under extended local gauge transformations, allowing the existence of two isomorphic representations of the intrinsic field algebra. The expected absence of θ-vacuum in the anomalous model [5] is displayed from the operator solution. We show the isomorphism between the gauge-noinvariant and gauge invariant field algebras. The role played by the Wess-Zumino (WZ) field in the anomalous chiral model with massive Fermi fields is the same played in the corresponding massless model. The introduction of the WZ field replicates the theory, changing neither its algebraic structure nor its physical content. This also streamlines the presentation of Refs. [5,12]. In order to obtain some insight into the fermion-boson mapping of the corresponding non-Abelian models, we shall consider the bosonization method using the Abelian reduction of the Wess-Zumino-Witten theory [6,17,18].

II. MASSIVE QED_2 WITH QUARTIC SELF-INTERACTION

The two-dimensional quantum electrodynamics with quartic self-interaction among massive fermions (massive Schwinger-Thirring model) is defined by the formal Lagrangian [13],

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{\mu\nu}^2 + \bar{\psi}(i\partial\!\!\!/ - e\mathcal{A}\!\!\!/ - M_o)\psi - \frac{g^2}{2}\, J^\mu\, J_\mu\,, \tag{2.1}$$

where J^μ is the fermionic current [1], $J^\mu = \bar{\psi}\gamma^\mu\psi$, and the field-strength tensor is $\mathcal{F}_{\nu\mu} \doteq \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$.

Within the operator formulation, the intrinsic local field algebra $\Im$ defining the model is generated from the set of local field operators $\{\bar{\psi}, \psi, \mathcal{A}_\mu\}$, such that, $\Im = \Im\{\bar{\psi}, \psi, \mathcal{A}_\mu\}$. From the vacuum expectation values of products of fields $\Phi(x) \in \Im$, we obtain the general Wightman functions $W^{(n)}$ of the theory

$$W^{(n)}(x_1, \ldots, x_n) = \langle\Omega|\Phi(x_1)\cdots\Phi(x_n)|\Omega\rangle\,, \tag{2.2}$$

where Ω denotes the state vector of the vacuum. From the Wightman functions $W^{(n)}$ the Hilbert space $\mathcal{H}$ can be constructed. The physical content of the model is given by the local gauge invariant field subalgebra $\Im_{gi} \subset \Im$, which is represented in the gauge invariant subspace $\mathcal{H}_{gi} \subset \mathcal{H}$, and built from the gauge invariant Wightman functions.

In the functional integral formulation of quantum field theory, the Euclidian Green functions are given by the averages

$$G^{(n)}(x_1, \cdots, x_n) = \frac{\int\, \Phi(x_1)\cdots\Phi(x_n)d\,\mu\,(\Phi)}{\int\, d\,\mu\,(\Phi)}\,, \tag{2.3}$$

where $d\,\mu\,(\Phi)$ is the probability measure in terms of the Euclidian action $S(\Phi)$,

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[1] Our conventions are: $x^\pm = x^0 \pm x^1$; $\partial_\pm = \partial_0 \pm \partial_1$; $\mathcal{A}^\pm = \mathcal{A}^0 \pm \mathcal{A}^1$;

$$g^{00} = 1 = -g^{11}; \epsilon^{01} = -\epsilon^{10} = 1; \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \gamma^5 = \gamma^0\gamma^1; \gamma^\mu\gamma^5 = \epsilon^{\mu\nu}\gamma_\nu\,,\ \tilde{\partial}_\mu = \epsilon_{\mu\nu}\partial^\nu\,.$$

The scalar and pseudoscalar massless free fields are decomposed as $\phi(x) = \phi(x^+) + \phi(x^-)$, $\phi(x) = \phi(x^+) - \phi(x^-)$, such that $\partial_\mu\,\phi(x) = \epsilon_{\mu\nu}\,\partial^\nu\,\phi(x)$.

$$d\mu(\Phi) = e^{\,-\,S[\Phi]}\,\mathcal{D}\Phi\,. \tag{2.4}$$

For Green functions satisfying the Osterwalder-Schrader axioms [14], the reconstruction theorem guarantees that the functions $\{G^{(n)}\}$ uniquely determine real-time Wightman distributions by analytic continuation in the time variables.

In what follows we shall consider two-dimensional gauge and anomalous gauge models using functional integral bosonization [2] and making a connexion with the operator formulation. In the present approach, which interpolates between the functional integral and operator approaches, close attention is paid to mantaining complete control on the Hilbert space structure needed for the representation of the intrinsic field algebra generated by the set of fundamental fields $\{\bar\psi, \psi, \mathcal{A}_\mu\}$ whose Wightman functions define the model.

A. Functional integral bosonization

To begin with, consider the generating functional in terms of the external Grassmann valued sources υ, $\bar\upsilon$, and the vector source ω_μ,

$$\mathcal{Z}[\upsilon,\bar\upsilon,\omega_\mu] = \langle\, e^{\,i\,(\,\bar\psi\,\upsilon\,+\,\bar\upsilon\,\psi\,+\,\mathcal{A}_\mu\,\omega^\mu\,)}\,\rangle_\Omega \;=\; \mathcal{N}^{-1}\int d\mu\; e^{\,i\int d^2z\,(\bar\psi\upsilon\,+\,\bar\upsilon\psi\,+\,\mathcal{A}_\mu\omega^\mu)}\,, \tag{2.5}$$

where the functional integral measure is given by

$$d\mu \;=\; \mathcal{D}\mathcal{A}_\mu\,\mathcal{D}\,\bar\psi\,\mathcal{D}\,\psi\;e^{\,i\,S[\bar\psi,\,\psi,\,\mathcal{A}_\mu]}\,, \tag{2.6}$$

and $S[\bar\psi,\psi,\mathcal{A}_\mu] = \int\, d^2x\,\mathcal{L}$, is the action corresponding to (2.1).

Due to the presence of the quartic Fermi field interaction, the first step in the standard procedure of the functional integral bosonization [1,3,9,11] is performed with the help of an "auxiliary" vector field b_μ. As stressed in Ref. [3], the introduction of the auxiliary vector field defines an

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[2]We shall consider the functional integral in Minkowski space and analytic continuation to Euclidian field theory is understood everywhere.

enlarged field algebra $\Im'$, defined by $\Im' = \Im'\{\bar{\psi}, \psi, \mathcal{A}_\mu, b_\mu\}$, such that $\Im \subset \Im'$. The field algebra $\Im'$ is represented in the Hilbert space $\mathcal{H}' = \Im'\Omega \supset \mathcal{H}$. This is donne by defining a new functional integral measure

$$d\mu' = d\mu\, \mathcal{D}\, b_\mu\, e^{i \int d^2x\, \frac{1}{2}\, b^\mu b_\mu}\,. \tag{2.7}$$

Introducing the source term for the "auxiliary" vector field, the Hilbert space $\mathcal{H}'$ can be built from the "interpolating" generating functional

$$\mathcal{Z}'[\upsilon, \bar{\upsilon}, \omega_\mu, \rho_\mu] = \langle\, e^{i \int d^2z\, (\, \bar{\psi}\, \upsilon + \bar{\upsilon}\, \psi + \mathcal{A}_\mu\, \omega^\mu \,+\, b_\mu \rho^\mu\,)}\, \rangle'_\Omega \,=$$
$$= \mathcal{N}^{-1} \int d\mu'\, e^{i \int d^2x\, (\bar{\psi}\upsilon + \bar{\upsilon}\psi + \mathcal{A}_\mu \omega^\mu + b^\mu \rho_\mu)}\,. \tag{2.8}$$

The source term for the field b_μ was included in order to control the effects of the auxiliary vector field on the bosonization procedure and the construction of the enlarged Hilbert space $\mathcal{H}'$.

In order to reduce the action of the Thirring model to a quadratic form in the Fermi field, the auxiliary vector field is shiffted by [1–3,9]

$$b_\mu = B_\mu - g\bar{\psi}\gamma_\mu\psi\,, \tag{2.9}$$

in a such way that

$$\int \mathcal{D}b_\mu\, e^{i \int d^2x\, \frac{1}{2}\, \{\, b^\mu b_\mu \,-\, g^2\, J^\mu\, J_\mu \}} = \int \mathcal{D}B_\mu\, e^{i \int d^2x\, \{\, \frac{1}{2}\, B_\mu\, B^\mu \,-\, g\, J^\mu\, B_\mu \}}\,. \tag{2.10}$$

This leads to a effective action in (2.8), which is given in terms of the Lagrangian density

$$\mathcal{L}'_{eff.} = -\frac{1}{4}\,\mathcal{F}^2_{\mu\nu} \,+\, \bar{\psi}\not{D}(\mathcal{A}, B)\psi - m_o\bar{\psi}\,\psi \,+\, \frac{1}{2}\, B_\mu\, B^\mu\,, \tag{2.11}$$

where the covariant derivative is defined by $\not{D}(\mathcal{A}, B) \doteq (i\not{\partial} - e\not{\mathcal{A}} \,-\, g\not{B})$. The local gauge invariance of the model emerges from gauge transformations acting on the local field subalgebra $\Im$,

$$\psi'(x) = e^{i\,\Lambda(x)}\,\psi(x)\,,$$
$$\mathcal{A}'_\mu(x) = \mathcal{A}_\mu(x) + \frac{1}{e}\,\partial_\mu\Lambda(x)\,. \tag{2.12}$$

The next step in the functional integral bosonization is to perform the decoupling of the Fermi and vector fields in the Lagrangian (2.11). To this end, we introduce the parametrization of the vector field components $(\mathcal{A}_\pm, B_\pm)$ in terms of the $U(1)$ group-valued Bose fields (U_a, V_a) and (U_b, V_b) as [6],

$$\mathcal{A}_+ = -\frac{1}{e}\, U_a^{-1}\, i\,\partial_+\, U_a \;\;,\;\; \mathcal{A}_- = -\frac{1}{e}\, V_a\, i\,\partial_-\, V_a^{-1}\,, \tag{2.13}$$

$$B_+ = -\frac{1}{g}\, U_b^{-1}\, i\,\partial_+\, U_b \;\;,\;\; B_- = -\frac{1}{g}\, V_b\, i\,\partial_-\, V_b^{-1}\,. \tag{2.14}$$

The decoupling is performed by the (Abelian) fermion chiral rotation [3,6],

$$\psi = \mathcal{M}\,\chi\,, \tag{2.15}$$

where the chiral rotation matrix $\mathcal{M}$ is given by

$$\mathcal{M} = \frac{1}{2}\,(1+\gamma^5)[U_aU_b]^{-1} + \frac{1}{2}\,(1-\gamma^5)V_aV_b\,, \tag{2.16}$$

in such way that

$$\bar{\psi}\not{D}(\mathcal{A},B)\psi \;=\; \chi^\dagger_{(1)}\, i\,\partial_-\,\chi_{(1)} \;+\; \chi^\dagger_{(2)}\, i\,\partial_+\,\chi_{(2)}\,. \tag{2.17}$$

Introducing in the functional integral the identities,

$$\begin{aligned} 1 &= \int \mathcal{D}(U_aU_b)\;\det\left(i\partial_+ - e\mathcal{A}_+ - gB_+\right)\delta\Big(e\mathcal{A}_+ + gB_+ - (U_aU_b)^{-1}\, i\,\partial_+\,(U_aU_b)\Big)\,,\\ 1 &= \int \mathcal{D}(V_aV_b)\;\det\left(i\partial_- - e\mathcal{A}_- - gB_-\right)\delta\Big(e\mathcal{A}_- + gB_- - (V_aV_b)\, i\,\partial_-\,(V_aV_b)^{-1}\Big)\,, \end{aligned} \tag{2.18}$$

the change of variables from $(\mathcal{A}_+, \mathcal{A}_-)$ to (U_a, V_a) and (B_+, B_-) to (U_b, V_b), is performed integrating over the vector fields components $(\mathcal{A}_\pm, B_\pm)$. Performing the fermion chiral rotation (2.15) and taking into account the corresponding change in the integration measure [3,6], we get

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi\,\mathcal{D}\mathcal{A}_\pm\,B_\pm = \mathcal{D}\bar{\chi}\mathcal{D}\chi\,\mathcal{D}U_a\,\mathcal{D}U_b\,\mathcal{D}V_a\,\mathcal{D}V_b\,\mathcal{J}\,[U_a, U_b, V_a, V_b]\,, \tag{2.19}$$

with

$$\mathcal{J} \;=\; e^{\,-i\,\left(\Gamma[U_aU_b] + \Gamma[V_aV_b] \;+\; c\,\int\, d^2z\,[(U_aU_b)^{-1}\partial_+(U_aU_b)]\,[(V_aV_b)\partial_-(V_aV_b)^{-1}]\right)}\,, \tag{2.20}$$

where $\Gamma[G]$ is the Wess-Zumino-Witten (WZW) functional [17], which enters in (2.20) with negative level. In the abelian case the WZW functional reduces to the free action

$$\Gamma[G] = \Gamma[G^{-1}] = \frac{1}{8\pi}\int d^2z\, \partial_\mu G \partial^\mu G^{-1}\,. \tag{2.21}$$

Using the Abelian Polyakov-Wiegman identity [18],

$$\Gamma[UV] = \Gamma[U] + \Gamma[V] + \frac{1}{4\pi}\int d^2z\, \big(U^{-1}\partial_+ U\big)\big(V\partial_- V^{-1}\big)\,, \tag{2.22}$$

and the gauge invariant regularization $c = 1/4\pi$, we can write (2.20) in terms of the gauge invariant Bose fields $G_\alpha \doteq U_\alpha V_\alpha$ ($\alpha = a, b.$),

$$\mathcal{J} = e^{-i\,\Gamma[G_a G_b]}\,. \tag{2.23}$$

The Maxwell Lagrangian can be written as

$$-\frac{1}{4}\mathcal{F}^2_{\mu\nu} = \frac{1}{8e^2}\Big\{\partial_+\Big(G_a i\partial_- G_a^{-1}\Big)\Big\}^2\,, \tag{2.24}$$

and the total effective action is given by

$$S'_{eff.} = -\Gamma[G_a G_b] + \int d^2z\,\Bigg\{\frac{1}{8e^2}\Big[\partial_+\Big(G_a i\partial_- G_a^{-1}\Big)\Big]^2 - \frac{1}{2g^2}\,\Big(U_b^{-1}\partial_+ U_b\Big)\Big(V_b\partial_- V_b^{-1}\Big) +$$

$$+\,\bar\chi\, i\,\partial\!\!\!/\,\chi - M_o\Big(\chi^*_{(1)}\chi_{(2)}\,(G_a G_b)^{-1} + \chi^*_{(2)}\chi_{(1)}\,(G_a G_b)\Big)\Bigg\} \tag{2.25}$$

The vector fields in two-dimensions can be decomposed as

$$\mathcal{A}_\mu = \frac{\sqrt{\pi}}{e}\,(\epsilon_{\mu\nu}\,\partial^\nu\,\phi_a + \partial_\mu \xi_a)\,, \tag{2.26}$$

$$B_\mu = \frac{\sqrt{\pi}}{g}\,(\epsilon_{\mu\nu}\,\partial^\nu\,\phi_b + \partial_\mu \xi_b)\,, \tag{2.27}$$

which corresponds to parametrizing the Bose fields (U_α, V_α) as follows,

$$U_\alpha = e^{\,i\,\sqrt{\pi}\,(\phi_\alpha + \xi_\alpha)}\,, \tag{2.28}$$

$$V_\alpha = e^{\,i\,\sqrt{\pi}\,(\phi_\alpha - \xi_\alpha)}\,. \qquad (2.29)$$

The effective Lagrangian density, corresponding to the action (2.25), can be written as

$$\mathcal{L}'_{eff.} = \frac{\pi}{2g^2}\,(\partial_\mu \xi_b)^2 \,+\, \frac{1}{2m^2}\,(\Box\phi_a)^2 \,+\, \frac{1}{2}\phi_a\Box\phi_a + \frac{1}{2}(1+\frac{\pi}{g^2})\phi_b\Box\phi_b + \phi_a\Box\phi_b +$$

$$+\bar{\chi}\,i\,\partial\!\!\!/\,\chi \,-\, M_o\,(\chi^*_{(1)}\chi_{(2)}\,e^{-2i\,\sqrt{\pi}\,(\phi_a + \phi_b)} + \text{h. c.})\,, \qquad (2.30)$$

where $m^2 = e^2/\pi$, is the mass of the gauge field in the absence of the Thirring coupling ($g = 0$, standard QED_2). Due to the local gauge invariance of the model, the field ξ_a does not appears in the effective Lagrangian (2.30). The field ξ_a is an unphysical pure gauge excitation, that appears only in the gauge non-invariant field operators ψ and $\mathcal{A}_\mu$, and can be gauged away from the operator solution (2.13-2.15). However, as we shall see, in the anomalous chiral model the field ξ_a acquires a non trivial dynamics and plays an important role in the algebraic structure of the model.

Defining the canonical field

$$\phi'_b = \frac{\sqrt{\pi}}{g}(1+\frac{g^2}{\pi})^{1/2}\phi_b = \frac{\sqrt{\pi}}{g}\alpha^{-1/2}\phi_b\,, \qquad (2.31)$$

the fields (ϕ_a, ϕ_b) can be decoupled using the identity

$$\phi'_b\,\Box\,\phi'_b \,+\, 2\,\frac{g}{\sqrt{\pi}}\,\alpha^{1/2}\,(\Box\,\phi_a)\,\phi'_b \,=$$

$$= (\phi'_b + \frac{g}{\sqrt{\pi}}\,\alpha^{1/2}\,\phi_a)\,\Box\,(\phi'_b + \frac{g}{\sqrt{\pi}}\,\alpha^{1/2}\,\phi_a) \,-\, \frac{g^2}{\pi}\,\alpha\,\phi_a\,\Box\,\phi_a \,=$$

$$= \,\eta_b\,\Box\,\eta_b \,-\, \frac{g^2}{\pi}\,\alpha\,\phi_a\Box\phi_a\,, \qquad (2.32)$$

where we defined the new field

$$\eta_b = \phi'_b + \frac{g}{\sqrt{\pi}}\,\alpha^{1/2}\,\phi_a\,. \qquad (2.33)$$

Introducing the parameter

$$\beta^2 = \frac{4\pi}{(1+\frac{g^2}{\pi})} = 4\pi\alpha\,, \tag{2.34}$$

and rescalling the fields

$$\phi'_a = \frac{\beta}{2\sqrt{\pi}}\,\phi_a \quad , \quad \xi'_b = \frac{\sqrt{\pi}}{g}\,\xi_b\,, \tag{2.35}$$

the effective Lagrangian density can be written as

$$\mathcal{L}'_{eff.} = \frac{1}{2}\,(\partial_\mu\xi'_b)^2 \, - \frac{1}{2}(\partial_\mu\eta_b)^2 \, + \, \frac{1}{2m_*^2}\,(\Box\phi'_a)^2 \, + \, \frac{1}{2}\phi'_a\Box\phi'_a +$$

$$+\bar{\chi}\,i\,\partial\!\!\!/\,\chi \, - \, M_o\,(\chi^*_{(1)}\chi_{(2)}\,e^{-\,i\,\beta\,(\frac{g}{\sqrt{\pi}}\,\eta_b\,+\,\phi'_a)} + \mathrm{h.\ c.}\,)\,, \tag{2.36}$$

where m_*^2 is the mass of the gauge field $\mathcal{A}_\mu$ in the presence of the Thirring coupling [13],

$$m_*^2 \, = \, m^2\,\frac{\beta^2}{4\pi} \, = \, \frac{e^2}{\pi + g^2}\,. \tag{2.37}$$

In order to decompose the higher derivative term in (2.36) in terms of a general local solution, we enlarge the Bose field algebra by using the functional integral identity

$$\int \mathcal{D}\phi'_a\, e^{i\int d^2z\Big\{\frac{1}{2m_*^2}\,\phi'_a\,\Box^2\,\phi'_a\,+\,\frac{1}{2}\,\phi'_a\Box\phi'_a\Big\}} \equiv$$

$$\equiv \int \mathcal{D}A \int \mathcal{D}\phi'_a\, e^{i\int d^2z\Big\{-\,\frac{1}{2}\,A^2\,+\,\frac{1}{m_*}(\Box A)\phi'_a\,+\,\frac{1}{2}\phi'_a\Box\phi'_a\Big\}} =$$

$$= \int \mathcal{D}\Sigma \int \mathcal{D}\eta_a\, e^{i\int d^2z\Big\{-\,\frac{1}{2}\,\Sigma\Big(\Box\,+\,m_*^2\Big)\Sigma\,-\,\frac{1}{2}(\partial_\mu\eta_a)^2\Big\}}\,, \tag{2.38}$$

where we have defined the fields

$$\eta_a = \phi'_a + \frac{1}{m_*}A \quad , \quad \Sigma = \frac{1}{m_*}A\,. \tag{2.39}$$

The effective Lagrangian (2.36) can be written as

$$\mathcal{L}'_{eff.} = \frac{1}{2}\,(\partial_\mu\xi'_b)^2 \, - \frac{1}{2}(\partial_\mu\eta_a)^2 \, - \frac{1}{2}(\partial_\mu\eta_b)^2 \, + \frac{1}{2}(\partial_\mu\Sigma)^2 \, - \frac{1}{2}m_*^2\Sigma^2 +$$

$$+\bar{\chi}\, i\, \partial\!\!\!/\, \chi \;-\; M_o \, (\chi^*_{(1)} \chi_{(2)} \, e^{-i\,\beta(\eta_a + \frac{g}{\sqrt{\pi}}\,\eta_b - \Sigma)} + h.c.\,)\,. \tag{2.40}$$

The last step is to perform the bosonization of the Fermi fields $\chi(x)$. In terms of the Bose fields, the original Fermi field operator $\psi(x)$ can be written as

$$\psi(x) \;=\, :e^{-\,i\,\frac{\beta}{2}\,\gamma^5\,\Sigma(x)}::e^{\,i\,\frac{\beta}{2}\,\gamma^5\,[\,\eta_a(x) + \frac{g}{\sqrt{\pi}}\,\eta_b(x)\,]}::e^{-\,i\,g\,\xi^\prime_b(x)}:\;\chi(x)\,. \tag{2.41}$$

The $2n$-point correlation functions of the Fermi fields are obtained by functional derivation of the generating functional with respect to the Grassmann valued sources $\bar{\upsilon}$ and υ, and can be written as

$$\langle\Omega|\,\bar{\psi}(x_1)\,\cdots\,\bar{\psi}(x_n)\,\psi(y_1)\,\cdots\,\psi(y_n)|\Omega\rangle = \langle\Omega|\prod_{i=1}^{n}\,e^{\,i\,g\,\xi^\prime_b(x_i)}\,\prod_{j=1}^{n}\,e^{-\,i\,g\,\xi^\prime_b(y_j)}\,|\Omega\rangle_o \times$$

$$\langle\Omega|\prod_{i=1}^{n}\,\bar{\chi}(x_i)\,e^{i\frac{\beta}{2}\gamma^5[\,\eta_a(x_i)+\frac{g}{\sqrt{\pi}}\,\eta_b(x_i)-\Sigma(x_i)]}\,\prod_{j=1}^{n}\,\chi(y_j)\,e^{-i\frac{\beta}{2}\gamma^5[\,\eta_a(y_j)+\frac{g}{\sqrt{\pi}}\,\eta_b(y_j)-\Sigma(y_j)]}\,|\Omega\rangle_I\,, \tag{2.42}$$

where the notation $\langle\Omega|\cdots|\Omega\rangle_o$ means average with respect to the free massless $\xi^\prime_b$-field theory, with the functional integral measure

$$d\mu(\xi^\prime_b) = \mathcal{D}\,\xi^\prime_b\,e^{\,i\,S_o[\xi^\prime_b]}\,, \tag{2.43}$$

and $\langle\Omega|\cdots|\Omega\rangle_I$ means average with respect to the effective Lagrangian $\mathcal{L}_I[\bar{\chi},\chi,\eta_a,\eta_b,\Sigma]$, with the measure,

$$d\mu_{_I} = \mathcal{D}\,\bar{\chi}\,\mathcal{D}\,\chi\,\mathcal{D}\,\eta_a\,\mathcal{D}\,\eta_b\,\mathcal{D}\,\Sigma\,e^{\,i\,\int\,d^2z\,\mathcal{L}_I[\bar{\chi},\chi,\eta_a,\eta_b,\Sigma]}\,. \tag{2.44}$$

Following the standard procedure [3,11,16], we perform the expansion of the exponential of the interaction term of $\mathcal{L}_I$ in a power series of the bare mass M_o, which provides the functional integral Gell-Mann and Low formula,

$$d\mu_{_I} = \sum_{n=0}^{\infty}\,(-iM_o)^n\frac{1}{n!}\,\prod_{j=1}^{n}\,\int d^2z_j\,\mathcal{D}\,\bar{\chi}\,\mathcal{D}\,\chi\,\mathcal{D}\,\eta_a\,\mathcal{D}\,\eta_b\,\mathcal{D}\,\Sigma\,e^{\,i\,S_o[\Sigma]}\,e^{-\,i\,S_o[\eta_a]}\,e^{-\,i\,S_o[\eta_b]}\,e^{\,i\,S_o[\bar{\chi},\chi]}\,\times$$

$$\prod_{k=1}^{n} \left[\chi^*_{(1)}(z_k)\chi_{(2)}(z_k)\, e^{-i\,\beta\{\eta_a(z_k) + \frac{g}{\sqrt{\pi}}\,\eta_b(z_k) - \Sigma(z_k)\}} + \text{h. c.}\right]. \tag{2.45}$$

The resulting correlation function, besides the contributions of the fields η_a, η_b and Σ, corresponds to averages of products of chiral density operators $\bar{\chi}(x_i)\chi(y_j)$ with respect to the free massless Fermi theory. The effective bosonized theory can be obtained by reconstructing the series using the free field bosonization expressions

$$\chi(x) = \Big(\frac{\mu_o}{2\pi}\Big)^{1/2} e^{-i\frac{\pi}{4}\gamma^5} : e^{i\sqrt{\pi}\{\gamma^5\varphi(x) + \int_{x^1}^{+\infty}\dot{\varphi}(x^0, z^1)\, dz^1\}} :, \tag{2.46}$$

$$\bar{\chi}\, i\,\partial\!\!\!/\,\chi \equiv \frac{1}{2} : (\partial_\mu\varphi)^2 :, \tag{2.47}$$

$$\chi^*_{(1)}\chi_{(2)} \equiv \frac{\mu_o}{2\pi} : e^{2i\sqrt{\pi}\varphi} :, \tag{2.48}$$

where the double dots indicate normal ordering with respect to the free propagator $(\Box + \mu_o^2)^{-1}$ in the limit $\mu_o \to 0$. In this way, the effective bosonized Lagrangian density is given by

$$\begin{aligned}\mathcal{L}'_{eff.} &= \frac{1}{2}(\partial_\mu\xi'_b)^2 - \frac{1}{2}(\partial_\mu\eta_a)^2 - \frac{1}{2}(\partial_\mu\eta_b)^2 + \frac{1}{2}(\partial_\mu\Sigma)^2 - \frac{1}{2}m_*^2\Sigma^2 + \\ &+ \frac{1}{2}(\partial_\mu\varphi)^2 + M\cos\{2\sqrt{\pi}\varphi - \beta(\eta_a + \frac{g}{\sqrt{\pi}}\eta_b - \Sigma)\}\,. \end{aligned} \tag{2.49}$$

We introduce two independent fields Φ and ζ, through the canonical transformation,

$$\gamma\Phi = 2\sqrt{\pi}\varphi - \frac{g}{\sqrt{\pi}}\,\beta\,\eta_b\,, \tag{2.50}$$

$$\gamma\zeta = \frac{g}{\sqrt{\pi}}\beta\varphi - 2\sqrt{\pi}\,\eta_b\,, \tag{2.51}$$

with

$$\gamma^2 = 4\pi - \frac{g^2}{\pi}\beta^2 = \beta^2 \tag{2.52}$$

and the field ζ is quantized with negative metric. In terms of these new fields, the effective bosonized Lagrangian density can be written as

$$\mathcal{L}_{eff.} = \frac{1}{2}\,(\partial_\mu \xi'_b)^2 \,-\, \frac{1}{2}(\partial_\mu \zeta)^2 \,-\, \frac{1}{2}(\partial_\mu \eta_a)^2 \,+\, \frac{1}{2}(\partial_\mu \Phi)^2 +$$

$$+\,\frac{1}{2}(\partial_\mu \Sigma)^2 \,-\, \frac{1}{2} m_*^2 \Sigma^2 \,+\, M\cos\{\beta\Sigma + \beta(\Phi \,-\, \eta_a)\}\,. \tag{2.53}$$

The effective bosonized theory is described by three coupled sine-Gordon-like fields Σ, Φ and η_a, with η_a quantized with negative metric, and two "decoupled" free massless fields ξ'_b and ζ, quantized with opposite metric. The field Σ is a massive sine-Gordon field with mass $m_*^2 = m^2\beta^2/4\pi$. The local gauge invariance of the model ensures that the Lagrangian (2.53) carries no dependence on the Bose field ξ_a, which is a pure gauge excitation. However, as we shall see, in the anomalous model the field ξ_a plays an important role in the field algebra.

B. Field algebra and Hilbert space

The introduction of the auxiliary vector field b_μ in the functional integral bosonization leads to an enlarged Bose field algebra containing two "decoupled" massless fields ξ'_b and ζ, quantized with opposite metric. As a matter of fact, the extraction of these decoupled massless Bose fields relies on a structural problem which is related to the fact that the fields ξ'_b and ζ do not belong to the field algebra $\Im'$ and cannot be defined by itself as operators in the Hilbert space $\mathcal{H}'$ [3–6]. We shall return to this point further on.

In order to display the structure of the Hilbert spaces $\mathcal{H}'$ and $\mathcal{H}$, it is instructive to express the fields $\{\bar\psi, \psi, \mathcal{A}_\mu, B_\mu\}$, and the corresponding source terms in the generating functional (2.8), in terms of the set of Bose fields $\{\Sigma, \Phi, \eta_a, \xi'_b, \zeta\}$. Using the decomposition (2.27) for the auxiliary vector field B_μ, and the corresponding Bose field transformations, we get

$$B_\mu\,(=\,gJ_\mu \,+\, b_\mu) \,=\, -\,g\,\frac{\beta}{2\pi}\,\epsilon_{\mu\nu}\partial^\nu \Big\{\,\Sigma \,+\, \Phi \,-\, \eta_a\,\Big\} \,+\, \partial_\mu\,(\,\xi'_b - \zeta\,)\,. \tag{2.54}$$

In this way, the vector current J^μ is identified as being given by

$$J_\mu \,=\, -\,\frac{\beta}{2\pi}\,\epsilon_{\mu\nu}\,\partial^\nu\,\Sigma \,+\, L_\mu\,, \tag{2.55}$$

where

$$L_\mu \;=\; -\,\frac{\beta}{2\pi}\,\epsilon_{\mu\nu}\partial^\nu(\Phi\,-\,\eta_a)\;\equiv\;\epsilon_{\mu\nu}\partial^\nu L\,, \tag{2.56}$$

is a longitudinal current. Since the sine-Gordon fields Φ and η_a are quantized with opposite metric, the corresponding equations of motion are given by

$$\Box\Phi-\beta\,M\;:\sin\{\beta\Sigma+\beta(\Phi\,-\,\eta_a)\}:=0\,, \tag{2.57}$$

$$-\,\Box\eta_a+\beta\,M\;:\sin\{\beta\Sigma+\beta(\Phi\,-\,\eta_a)\}:=0\,. \tag{2.58}$$

In this way, the field combination $L=(\Phi-\eta_a)$, is a free massless field

$$\Box(\Phi-\eta_a)=0\,, \tag{2.59}$$

which generates zero norm states from the vacuum,

$$\langle\Omega|\,\Big(\Phi(x)-\eta_a(x)\Big)\Big(\Phi(y)-\eta_a(y)\Big)\,|\Omega\rangle=0\,. \tag{2.60}$$

The auxiliary vector field b_μ is identified with a longitudinal current ℓ_μ, given in terms of the two free and massless Bose fields quantized with opposite metric

$$b_\mu\equiv\ell_\mu=\;\partial_\mu(\xi^\prime_b-\zeta)\,, \tag{2.61}$$

such that

$$\langle\Omega|\ell_\mu(x)\,\ell_\mu(y)\,|\Omega\rangle^\prime=0\,. \tag{2.62}$$

In the limit $g=0$, the vector field B_μ is a longitudinal field $B_\mu\equiv\ell_\mu$.

In a similar way, the gauge field (2.26) can be written as

$$\mathcal{A}_\mu=-\,\frac{1}{m_*}\,\epsilon_{\mu\nu}\partial^\nu(\Sigma-\eta_a)\,. \tag{2.63}$$

Performing the fermion chiral rotation (2.15), together with the canonical transformations (2.51), and using the bosonized expression (2.46) for the free massive Fermi field, we obtain the operator solution for the Fermi field $\psi(x)$ of the massive Schwinger-Thirring model in terms of the Mandelstam [19] "soliton" field operator as

$$\psi(x) = \Psi(x)\,\Upsilon(x)\,, \tag{2.64}$$

where Ψ is the operator solution of the massive Schwinger-Thirring model

$$\Psi(x) \,=: e^{\,i\,\frac{\beta}{2}\gamma^5\,\{\Sigma(x)\,-\,\eta_a(x)\,\}}\, :\, \mathcal{S}(x)\,, \tag{2.65}$$

where $\mathcal{S}(x)$ is the Mandelstam representation for the massive Fermi field operator of the Thirring model [19]

$$\mathcal{S}(x) = \Big(\frac{\mu_o}{2\pi}\Big)^{1/2} e^{-\,i\,\frac{\pi}{4}\gamma^5}\, :e^{\,i\,\frac{\beta}{2}\gamma^5\,\Phi(x)\,+\,\frac{2\pi}{\beta}\,i\,\int_{x^1}^{\infty}\,\dot{\Phi}(x^0,z^1)dz^1}\,:\,. \tag{2.66}$$

In Eq. (2.64), both spinor components of the Fermi field $\Psi_{(\alpha)}(x)$ are multiplied by the exponential field

$$\Upsilon(x) \,=: e^{-\,i\,g\,\{\,\xi_b'(x)\,-\,\zeta(x)\,\}}\,:=: e^{\,ig\,\ell(z)}\,:, \tag{2.67}$$

where $\ell = \xi_b' - \zeta$, is the potential for the longitudinal current (2.61).

Besides the electric field and the vector current, the gauge invariant field subalgebra is generated by the bilocal operators formally defined by [20–24]

$$D_{\alpha\beta}(x,y) \approx \psi_\alpha^*(x)\,e^{\,ie\int_y^x \mathcal{A}_\mu(z)dz^\mu}\,\psi_\beta(y)\,. \tag{2.68}$$

The gauge invariant composite operator (2.68) can be factorized as

$$D_{\alpha\beta}(x,y) \,=\, T_{\alpha\beta}(x,y)\,\sigma_\alpha^*(x)\sigma_\beta(y)\,W_{_b}(x,y)\,, \tag{2.69}$$

where $T_{\alpha\beta}(x,y)$ is the generalization of the bilocal operator obtained in [21] for the massive QED_2,

$$T_{\alpha\beta}(x,y) \,=\, N(x-y)\,:e^{-\,i\,\frac{\beta}{2}\,\{\,\gamma^5_{\alpha\alpha}\,\Sigma(x)\,-\,\gamma^5_{\beta\beta}\,\Sigma(y)\,\}\,-\,\frac{2\pi}{\beta}\,i\,\int_y^x\,\epsilon_{\mu\nu}\partial^\nu\Sigma(z)dz^\mu}\,:, \tag{2.70}$$

$\sigma_\alpha(x)$ is the spurious operator introduced by Lowenstein and Swieca [20–22],

$$\sigma_\alpha(x) \,=: e^{\,i\,\frac{\beta}{2}\,\gamma^5_{\alpha\alpha}\,\{\,\Phi(x)\,-\,\eta_a(x)\,\}\,-\,\frac{2\pi}{\beta}\,i\,\int_{x^1}^{\infty}\,\{\,\dot{\Phi}(z)\,-\,\dot{\eta}_a(z)\}\,dz^1}\,:, \tag{2.71}$$

and $W_{_b}[x,y]$ is a spurious "bilocal"operator given by

$$W_{_b}[x,y] \doteq : e^{ig\int_{x,C}^{y}\ell_\mu(z)dz^\mu} : , \tag{2.72}$$

where C is an arbitrary curve.

The chiral density operator is computed from the point-splitting limit procedure of the bilocal operator (2.68), and we get

$$\mathcal{J}(x) \doteq \vdots\psi^*_{_{(1)}}(x)\,\psi_{_{(2)}}(x)\vdots =: e^{-i\beta\,\Sigma(x)} : \sigma^*_{_{(1)}}(x)\sigma_{_{(2)}}(x)\,, \tag{2.73}$$

where

$$\sigma^*_{_{(1)}}(x)\,\sigma_{_{(2)}}(x) =: e^{i\beta[\Phi(x)-\eta_a(x)]} : , \tag{2.74}$$

is a spurious operator carrying the free fermion chirality [20–23].

The interpolating generating functional (2.8), from which we construct the Hilbert space $\mathcal{H}'$, can be written in terms of the Bose fields as

$$\mathcal{Z}'[\upsilon,\bar\upsilon,\omega_\mu,\rho_\mu] = \langle e^{i\int d^2z\,(\bar\psi\upsilon+\bar\upsilon\psi+\mathcal{A}_\mu\omega^\mu+b_\mu\rho^\mu)}\rangle' =$$

$$= \langle e^{i\int d^2z\,(\overline{\Psi}\Upsilon^*\upsilon+\bar\upsilon\Psi\Upsilon+\frac{1}{m_*}\,\omega^\mu\epsilon_{\mu\nu}\partial^\nu(\Sigma-\eta_a)\;+\;\rho^\mu\partial_\mu[\xi'_b-\zeta])}\rangle' \tag{2.75}$$

where the average is taken with respect to the measure

$$d\mu' = \mathcal{N}^{-1}\int\mathcal{D}\xi'_b\,e^{\,i\,S_o[\xi'_b]}\int\mathcal{D}\zeta\,e^{\,-i\,S_o[\zeta]}\int\mathcal{D}\Phi\int\mathcal{D}\eta_a\int\mathcal{D}\Sigma\;e^{i\,S\,[\Sigma,\,\Phi,\,\eta_a]}\,, \tag{2.76}$$

where the S_o's are the free actions for the massless fields quantized with opposite metric.

¿From the generating functional (2.75) we obtain the general Wightman $2n$-point functions for the Fermi field in terms of averages of order-disorder operators

$$\langle\Omega|\,\bar\psi(x_1)\,\cdots\,\bar\psi(x_n)\,\psi(y_1)\,\cdots\,\psi(y_n)|\Omega\rangle' =$$

$$\langle\Omega|\,\overline{\Psi}(x_1)\,\cdots\,\overline{\Psi}(x_n)\,\Psi(y_1)\,\cdots\,\Psi(y_n)\,|\Omega\rangle\langle\Omega|\Upsilon^*(x_1)\,\cdots\,\Upsilon^*(x_n)\,\Upsilon(y_1)\,\cdots\,\Upsilon(y_n)\,|\Omega\rangle_o\,, \tag{2.77}$$

where the notation $\langle\Omega|\cdots|\Omega\rangle$ means average with respect to coupled sine-Gordon fields Σ, Φ, η_a, and $\langle\Omega|\cdots|\Omega\rangle_o$ means average with respect to the free theories of the massless Bose fields ξ_b' and ζ. Due to the opposite metric quantization for the fields ζ and ξ_b', the functional integration over the field ξ_b' cancels those arising from the integration over the field ζ. This implies that the field $W_{_b}$ generates constant constributions to the Wightman functions,

$$\langle\Omega|\,\Upsilon^*(x_1)\,\cdots\,\Upsilon^*(x_n)\,\Upsilon(y_1)\,\cdots\,\Upsilon(y_n)\,|\Omega\rangle_o \equiv \langle\Omega|\prod_{j=1}^{n} : e^{\,ig\int_{x_j,C_j}^{y_j}\ell_\mu(z)dz^\mu} : |\Omega\rangle_o =$$

$$= \langle\Omega|\prod_{j=1}^{n} W_{_b}[x_j,y_j]|\Omega\rangle_o = 1\,. \tag{2.78}$$

The fact that the operator $W_{_b}[x,y]$ generates constant constributions in (2.77), implies the isomorphism between the Wightman functions of the Fermi field in the Hilbert spaces $\mathcal{H}'$ and $\mathcal{H}$:

$$\langle\Omega|\,\bar\psi(x_1)\,\cdots\,\bar\psi(x_n)\,\psi(y_1)\,\cdots\,\psi(y_n)|\Omega\rangle' \equiv \langle\Omega|\,\overline{\Psi}(x_1)\,\cdots\,\overline{\Psi}(x_n)\,\Psi(y_1)\,\cdots\,\Psi(y_n)\,|\Omega\rangle\,. \tag{2.79}$$

For any functional $\mathcal{F}\{\bar\psi,\psi\}\in\Im'$, we obtain the general one-to-one mapping in the Hilbert spaces $\mathcal{H}'$ and $\mathcal{H}$:

$$\langle\Omega|\,\mathcal{F}\{\bar\psi,\psi\}\,|\Omega\rangle' \equiv \langle\Omega|\,\mathcal{F}\{\overline{\Psi},\Psi\}\,|\Omega\rangle\,. \tag{2.80}$$

The enlarged Bose field algebra $\Im^{^B}$ contains two spurious fields with zero scale dimension, e. g., $\Upsilon(x)$ and

$$\hat\sigma(x) \doteq \sigma^*_{_{(1)}}(x)\,\sigma_{_{(2)}}(x)\,. \tag{2.81}$$

As is well known from the QED_2 with massless Fermi fields [21,23], the spurious operator $\hat\sigma(x)$ is the generator of the integer winding number gauge transformations in the physical Hilbert space. This will remains valid in the present model. However, the operator $\Upsilon(x)$ cannot be defined by itself in the field algebra $\Im'$ [3]. Nevertheless, the operator $W_{_b}(x,y)$ belongs to the field algebra $\Im'$, but it reduces to the identity in $\mathcal{H}'$. From the functional integral point of view, although the partition function obtained from (2.75) factorizes in the form

$$\mathcal{Z}'[0] = \mathcal{Z}^{o}_{\zeta}[0] \times \mathcal{Z}^{o}_{\xi'_b}[0] \times \mathcal{Z}_{\Sigma,\Phi,\eta a}[0]\,, \tag{2.82}$$

the fact that the " spurious " field Υ appears attached to the bosonized Fermi field Ψ in the source terms, suggests that the generating functional (2.75) cannot be factorized and the so-called "decoupled" massless scalar fields cannot be removed in a naive way, contrary to what is usually done [10,11]. The origin of this extra spurious field relies on the functional integral bosonization procedure that makes use of the introduction of the auxiliary vector field B_μ. This question can be clarified on the basis of the intrinsic algebraic structure of the model.

The set of fields $\{\bar{\psi}, \psi, \mathcal{A}_\mu\}$ constitute the intrinsic mathematical structure of the model and generate the local polynomial field algebra $\Im = \Im\{\bar{\psi}, \psi, \mathcal{A}_\mu\}$. The Wightman functions generated from the field algebra $\Im$ define the model and identifies the Hilbert space $\mathcal{H}$ of the theory, $\mathcal{H} \doteq \Im\Omega$. The introduction of the auxiliary vector field b_μ enlarge the field algebra $\Im \rightarrow \Im' = \Im'\{b_\mu, \bar{\psi}, \psi, \mathcal{A}_\mu\}$, and the change of variables (2.9) leads to a new field algebra $\Im' = \Im'\{\mathcal{B}_\mu, \bar{\psi}, \psi, \mathcal{A}_\mu\}$. This field algebra is represented in the enlarged Hilbert space $\mathcal{H}' \doteq \Im'\Omega$. Within the bosonization procedure the fundamental fields defining the field algebra $\Im'$ are written in terms of the Bose fields $\{\Sigma, \Phi, \eta_a, \zeta, \xi'_b\}$. This set of Bose fields define an enlarged redundant field algebra $\Im^{B}$, which is represented in the indefinite metric Hilbert space $\mathcal{H}^{B} \doteq \Im^{B}\Omega$. These Bose fields are the building blocks in terms of which the bosonized solution is constructed and, as stressed in Refs. [4–6], should not be considered as elements of the field algebra $\Im'$. Only some particular combinations of them belong to the field algebra $\Im'$, in such a way that, $\Im' \subset \Im^{B}$, and thus, $\mathcal{H}' \subset \mathcal{H}^{B}$. The auxiliary vector field $B_\mu = gJ_\mu + \ell_\mu$, belong to the field algebra $\Im'$ and since $J_\mu \in \Im'$, then, $\ell_\mu \in \Im'$. In this way, the positive semi-definite Hilbert space $\mathcal{H}'$ is generated from the field algebra $\Im'\{B_\mu, \bar{\psi}, \psi, \mathcal{A}_\mu\} = \Im'\{\Im'_o, \overline{\Psi}\Upsilon^*, \Psi\Upsilon, \mathcal{A}_\mu\}$, where $\Im'_o \subset \Im'$ is the field subalgebra generated by the longitudinal current ℓ_μ, $\Im'_o = \Im'_o\{\ell_\mu\}$, and that generates zero norm states: $\mathcal{H}'_o \doteq \Im'_o\Omega \subset \mathcal{H}'$. The field $\ell = (\zeta + \xi_b)$, that acts as potential for the longitudinal current ℓ_μ, does not belong to the field algebra $\Im'$ and only its space-time derivatives occur in $\Im'$. In this way, the exponential field Υ given by (2.67) also does not belong to $\Im'$ and the Hilbert space cannot be factorized, $\mathcal{H}' \neq \mathcal{H}_\Upsilon \otimes \mathcal{H}_\Psi$.

¿From the algebraic point of view, the fact that the field Υ does not belong to the field algebra $\Im'$ and thus cannot be defined as an operator in $\mathcal{H}'$, follows from the charge content of $\mathcal{H}_{_B}$ and $\mathcal{H}'$, since some *topological* charges get trivialized in going from $\mathcal{H}^{^B}$ to $\mathcal{H}$ [4]. One can define conserved currents j_μ belonging to the Bose field algebra $\Im^{^B}$, as for example,

$$j^\mu(x) \doteq \frac{\beta}{2\pi}\,\epsilon_{\mu\nu}\,\partial^\nu\Phi(x) + \partial^\mu\zeta \,\in\, \Im^{^B}\,. \tag{2.83}$$

The corresponding charges are

$$\mathcal{Q} \doteq \int_{-\infty}^{+\infty} dz^1\, j^0(z)\,, \tag{2.84}$$

such that

$$[\mathcal{Q}, \Im^{^B}] \neq 0\,. \tag{2.85}$$

This implies that the charges $\mathcal{Q}$ do not vanish on $\mathcal{H}^{^B}$:

$$\mathcal{Q}\mathcal{H}^{^B} \neq 0\,. \tag{2.86}$$

The charges $\mathcal{Q}$ commute with ψ, J^μ and ℓ_μ, that is,

$$[\mathcal{Q}, \Im_o] = 0 \;\;,\;\; [\mathcal{Q}, \Im'] = 0 \;\;\rightarrow\;\; [\mathcal{Q}, \Im] = 0\,. \tag{2.87}$$

This means that the charges $\mathcal{Q}$ are trivialized in the restriction from $\mathcal{H}^{^B}$ to $\mathcal{H}'$ or $\mathcal{H}$ [4–6]:

$$\mathcal{Q}\mathcal{H}^{^B} \neq 0 \;\;,\;\; \mathcal{Q}\mathcal{H}' = 0 \;\;,\;\; \mathcal{Q}\mathcal{H} = 0\,. \tag{2.88}$$

Since $[\mathcal{Q}, \Upsilon] = \alpha\Upsilon$, the state $\Upsilon\Omega$ cannot belong to $\mathcal{H}'$ and the field Υ cannot be defined as an operator in the Hilbert space $\mathcal{H}'$ [4–6]. Nevertheless, the spurious "bilocal"operator $W_{_b}(x,y)$ is neutral under $\mathcal{Q}$ and can be defined as an element of $\mathcal{H}'$ and leads to constant correlation functions

$$\langle\Omega|\, W_{_b}(x_1,y_1)\, W_{_b}(x_2,y_2)\cdots W_{_b}(x_n,y_n)\,|\Omega\rangle \;=\; 1\,. \tag{2.89}$$

The infinitely delocalized state $W_{_b}(x,y)\,\Omega$ is translationally invariant in $\mathcal{H}'$. The position independence of this state can be seen by computing the general Wightman functions involving the operator $W_{_b}(x,y)$ and all operators belonging to the local field algebra $\Im'$. For any operator

$$\mathcal{O}(f_z) = \int \mathcal{O}(z) f(z) d^2 z \in \Im' \,, \tag{2.90}$$

of polynomials in the smeared fields defining the field algebra $\Im'$, the position independence of the operator (3.71) can be expressed in the weak form as

$$\langle \Omega | \prod_{i=1}^{m} : e^{ig \int_{x_i}^{y_i} \ell_\mu(z) dz^\mu} : \mathcal{O}(f_{z_1}, \ldots, f_{z_n}) | \Omega \rangle \;=\; F(z_1, \ldots, z_n) \equiv \langle \Omega | \mathcal{O}(f_{z_1}, \ldots, f_{z_n}) | \Omega \rangle \,, \tag{2.91}$$

where $F(z_1, \ldots, z_n)$ is a distribution independent of the space-time coordinates (x_i, y_i). Since the operator $W_b(x,y)$ carries no selection rule, it reduces to the identity in $\mathcal{H}'$. By the other hand, the operator $\hat{\sigma}(x)$ does not reduces to the identity in $\mathcal{H}$ since it carries the fermion chirality. As is well known, this operator gives rises to the existence of an infinite degeneracy of the ground-state implying the θ-vacuum parametrization [20,21].

The current ℓ_μ commutes with itself and with all operators belonging to the field algebra $\Im'$,

$$[\ell_\mu(x), \Im'] = 0 \,, \tag{2.92}$$

implying that,

$$\langle \Theta' | \ell_\mu(x) | \Xi' \rangle = 0 = \langle \Theta' | \Big(B_\mu(x) - g \bar{\psi}(x) \gamma_\mu \psi(x) \Big) | \Xi' \rangle \;\;,\; \forall \, |\Theta'\rangle, |\Xi'\rangle \in \mathcal{H}' \,. \tag{2.93}$$

The states in $\mathcal{H}'$ can be accomodated as equivalence classes modulo ℓ_μ, in such a way that the Hilbert space $\mathcal{H}$ is the quocient space

$$\mathcal{H} = \frac{\mathcal{H}'}{\mathcal{H}'_o} \,. \tag{2.94}$$

¿From the operator point of view, the equivalence established by Eq. (2.79) implies the algebraic isomorphism

$$\Im\{\bar{\psi}, \psi, \mathcal{A}_\mu\} \sim \Im''\{\overline{\Psi}\Upsilon^*, \Psi\Upsilon, \mathcal{A}_\mu\} \sim \Im\{\overline{\Psi}, \Psi, \mathcal{A}_\mu\} \,, \tag{2.95}$$

where $\Im''\{\overline{\Psi}\Upsilon^*, \Psi\Upsilon, \mathcal{A}_\mu\} \subset \Im'\{\Im'_o, \overline{\Psi}\Upsilon^*, \Psi\Upsilon, \mathcal{A}_\mu\}$. Since in the quocient space $\mathcal{H}$, the current ℓ_μ is the null element, $\ell \equiv 0$, within the functional integral formalism the quocient space is obtained with $\rho_\mu = 0$. Taking into account the fact that the operator $W_b(x,y)$ becomes the identity in $\mathcal{H}$, we obtain the equivalence in the functional integral approach

$$\mathcal{Z}'[\upsilon,\bar{\upsilon},\omega_\mu,0] \sim \mathcal{Z}[\upsilon,\bar{\upsilon},\omega_\mu] \sim \mathcal{Z}_{\Phi,\Sigma,\eta_a}[\upsilon,\bar{\upsilon},\omega_\mu]\,, \tag{2.96}$$

where

$$\mathcal{Z}_{\Phi,\Sigma,\eta_a}[\upsilon,\bar{\upsilon},\omega_\mu] = \int \mathcal{D}\Phi\,\mathcal{D}\eta_a\,\mathcal{D}\Sigma\, e^{i\,S\,[\Sigma,\Phi,\eta_a]}\, e^{i\int d^2x\,\{\overline{\Psi}\,\upsilon + \bar{\upsilon}\,\Psi\,+\,\frac{1}{m_*}\,\omega^\mu\epsilon_{\mu\nu}\partial^\nu(\Sigma-\eta_a)\,\}}\,, \tag{2.97}$$

is the generating functional of the Schwinger-Thirring model with massive fermions. This establishes the fermion-boson mapping between the massive Schwinger-Thirring model and the coupled sine-Gordon theories in the positive semi-definite Hilbert space $\mathcal{H}$.

The states in $\mathcal{H}$ can be accomodated as equivalent classes modulo L_μ, in such a way that the positive-definite gauge invariant Hilbert space $\hat{\mathcal{H}}$ is the quocient space

$$\hat{\mathcal{H}} = \frac{\mathcal{H}}{\mathcal{H}_o}\,, \tag{2.98}$$

where $\mathcal{H}_o = L_\mu\,\Omega$.

The appearance in the mass term of the spurious field $\hat{\sigma}(x)$, that carry the free fermion field chirality, signalize the explicit breakdown of the chiral symmetry. For $M_o \rightarrow 0$, the fields $\Phi(x)$, $\eta_a(x)$ becomes free and massless, Σ is a massive free field, and we recover the massless Schwinger-Thirring model.

III. ANOMALOUS QED_2 WITH MASSIVE FERMIONS

In this section we shall consider the functional integral bosonization of the anomalous QED_2 with massive Fermi fields, defined from the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{\mu\nu}^2 + \chi_r^\dagger i\partial_+\chi_r\,+\,\psi_\ell^\dagger(i\partial_- - e\mathcal{A}_-)\psi_\ell - M_o(\chi^\dagger{}_r\psi_\ell + \psi_\ell^\dagger\chi_r)\,. \tag{3.1}$$

The anomalous model with massless Fermi fields was considered in detail in Refs. [5,12] using the operator approach. The so-called "decoupled" massless free scalar Bose fields play an important role in the bosonized massless chiral model in order to ensure the existence of Fermi fields in the asymptotic states, the cluster decomposition property of the Wightman functions, the inexistence of

θ-vacuum and the isomorphism between the gauge invariant and gauge non-invariant formulations [6,7]. In what follows we shall discuss the role played by these fields in the corresponding anomalous chiral model with massive fermions.

A. Functional integral bosonization

Following the same procedure of the previous section, we parametrize the gauge field in terms of the Bose fields $\{U, V\}$,

$$\mathcal{A}_+ = -\frac{1}{e} U^{-1} i\partial_+ U \,, \tag{3.2}$$

$$\mathcal{A}_- = -\frac{1}{e} V i\partial_- V^{-1} \,. \tag{3.3}$$

Introducing the decomposition for the gauge field [3]

$$\mathcal{A}_\mu = \frac{\sqrt{\pi}}{e} \{\epsilon_{\mu\nu}\partial^\nu \phi + \partial_\mu \xi\} \,, \tag{3.4}$$

we can write,

$$U = e^{\, i\sqrt{\pi}(\phi + \xi)} \,, \tag{3.5}$$

$$V = e^{\, i\sqrt{\pi}(\phi - \xi)} \,. \tag{3.6}$$

Performing the chiral fermion rotation

$$\psi_\ell = V \chi_\ell \,, \tag{3.7}$$

the Jacobian associated with the change in the functional integral measure is given by

$$\mathcal{J} = e^{\, -i\Gamma[V] + i\frac{a}{8\pi} \int d^2 z (U^{-1} i\partial_+ U)(V i\partial_- V^{-1})} \,. \tag{3.8}$$

---

[3]With the notation of the preceding section, $\phi \equiv \phi_a$, $\xi \equiv \xi_a$.

The presence of the last factor in (3.8) reflects the usual regularization ambiguity, with a the Jackiw-Rajaraman parameter [5,6,12,26].

Defining the gauge invariant field $G \doteq UV$, we can write the Jacobian (3.8) as

$$\mathcal{J} = e^{-i\frac{a}{2}\Gamma[G] + i(\frac{a}{2} - 1)\Gamma[V] + i\frac{a}{2}\Gamma[U]} . \tag{3.9}$$

The total effective action is given by

$$\begin{aligned} S_{eff.} &= S_M[G] - \frac{a}{2}\Gamma[G] + (\frac{a}{2} - 1)\Gamma[V] + \frac{a}{2}\Gamma[U] + \\ &+ \int d^2z \left\{ \bar{\chi}\, i\partial\!\!\!/\, \chi - M_o \left(\chi_r^* \chi_\ell\, V^{-1} + \chi_\ell^* \chi_r\, V \right\} \right. , \end{aligned} \tag{3.10}$$

where $S_M[G]$ is the Maxwell action,

$$S_M[G] = \frac{1}{8e^2} \int d^2z \left[\partial_+ \left(G\, i\, \partial_- G^{-1} \right) \right]^2 . \tag{3.11}$$

For $a = 2$ and $M_o = 0$, the field V decouples from (3.10) and except by the presence of the action $\Gamma[U]$, which appears to play merely a spectator role, the effective action (3.10) is that of the QED_2 with massless Fermi fields,

$$S_{eff.}\Big|_{\substack{a=0\\ M=0}} = S_{Sch} + \Gamma[U] . \tag{3.12}$$

As shown in Refs. [5,6], the apparently decoupled field U plays an important role in the construction of the Hilbert space of the anomalous chiral model with massless fermions.

Following the same procedure of the preceding section, performing the bosonization of the free massive Fermi field χ in terms of the Bose field φ, the effective Lagrangian density can be written as

$$\begin{aligned} \mathcal{L} &= \frac{4\pi}{2e^2}(\Box\phi)^2 + \frac{1}{2}(a+1)\phi\Box\phi - \frac{1}{2}\xi'\Box\xi' - (a-1)^{-1/2}(\Box\phi)\xi' + \\ &- \frac{1}{2}\varphi\Box\varphi - M_o' \cos\{2\sqrt{\pi}\varphi - 2\sqrt{\pi}(\phi - (a-1)^{-1/2}\xi')\} , \end{aligned} \tag{3.13}$$

where we have defined the canonical field

$$\xi' = (a-1)^{1/2}\xi \,. \tag{3.14}$$

The effect of the anomaly introduces the field ξ', which is a pure gauge excitation in the vector model, in the effective bosonized chiral model [5,6,12]. Due to the presence of a mass term for the Fermi fields, the field ξ' no longer is a free field. As we shall see, the field ξ' plays an important role in the field algebra and in the construction of the Hilbert space of the anomalous chiral model with massive fermions.

The fields ξ' and ϕ can be decoupled in (3.13) using that

$$-\frac{1}{2}\Big(\xi'\Box\xi' + 2(a-1)^{-1/2}(\Box\phi)\xi'\Big) = -\frac{1}{2}\xi''\Box\xi'' + \frac{1}{2}\phi'\Box\phi' \,, \tag{3.15}$$

where

$$\phi' = \frac{a}{\sqrt{a-1}}\,\phi \;\;,\;\; \xi'' = \xi' + \phi' \,. \tag{3.16}$$

The effective Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2m_a^2}(\Box\phi')^2 + \frac{1}{2}\phi'\Box\phi' - \frac{1}{2}\xi''\Box\xi'' - \frac{1}{2}\varphi\Box\varphi -$$

$$-M_o\cos\Big\{2\sqrt{\pi}\varphi + 2\sqrt{\frac{\pi}{a-1}}(\xi''-\phi')\Big\} \,, \tag{3.17}$$

where

$$m_a^2 = (e\,a)^2\,/\,4\pi(a-1) \,. \tag{3.18}$$

In a similar way, the higher-derivative term in (3.17) can be reduced using that

$$\int \mathcal{D}\phi'\, e^{\,i\int d^2z\big\{\frac{1}{2m_a^2}(\Box\phi')^2 + \frac{1}{2}\phi'\Box\phi'\big\}} \equiv$$

$$\int \mathcal{D}\Sigma \int \mathcal{D}\phi''\, e^{\,i\int d^2z\big\{-\frac{1}{2}\Sigma\Box\Sigma - \frac{1}{2}m_a^2\Sigma^2 + \phi''\Box\phi''\big\}} \,. \tag{3.19}$$

For further convenience we relabel the field $\phi'' = \eta$, and the effective bosonized Lagrangian density is then given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \xi'')^2 - \frac{1}{2}(\partial_\mu \eta)^2 + \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}(\partial_\mu \Sigma)^2 - \frac{1}{2} m_a^2 (\Sigma)^2 -$$
$$- M_o' : \cos\Big\{ 2\sqrt{\frac{\pi}{a-1}}\, \Sigma \,+\, 2\sqrt{\pi}\varphi \,+\, 2\sqrt{\frac{\pi}{a-1}}\, (\, \xi'' - \eta \,) \Big\} : \,. \tag{3.20}$$

The mass term carry dependence on the fields ξ'' and η, which are quantized with opposite metric, the massive field Σ and the field φ, which carries the free fermion degrees of freedom. Since the fields ξ'' and η are quantized with opposite metric, the corresponding equations of motion are given by

$$\Box \xi'' \,-\, 2\Big(\frac{\pi}{a-1}\Big)^{1/2} M_o' : \sin\Big\{ 2\sqrt{\frac{\pi}{a-1}}\, \Sigma \,+\, 2\sqrt{\pi}\varphi \,+\, 2\sqrt{\frac{\pi}{a-1}}\, (\, \xi'' - \eta \,) \Big\} := 0 \,, \tag{3.21}$$

$$- \,\Box \eta \,+\, 2\Big(\frac{\pi}{a-1}\Big)^{1/2} M_o' : \sin\Big\{ 2\sqrt{\frac{\pi}{a-1}}\, \Sigma \,+\, 2\sqrt{\pi}\varphi \,+\, 2\sqrt{\frac{\pi}{a-1}}\, (\, \xi'' - \eta \,) \Big\} := 0 \,. \tag{3.22}$$

Although the fields ξ'' and η acquire a non trivial dynamics, the field combination $(\xi'' \,-\, \eta)$ is a free and massless field

$$\Box (\xi'' \,-\, \eta) = 0 \,, \tag{3.23}$$

that generates zero norm states from the vacuum. For massless fermions ($M_o = 0$) the fields ξ'' and η, becomes massless free fields and even in this case we cannot disregard them from the field algebra [5].

B. Field algebra

The set of fields $\{\bar{\psi}, \psi, \mathcal{A}_\mu\}$ defines the field algebra $\Im$ and constitute the intrinsic mathematical description of the model. In terms of the Bose fields, the Fermi field can be written as

$$\psi(x) \;=: e^{\displaystyle i\,\sqrt{\pi}\, \frac{(1+\gamma^5)}{2}\, [\phi(x) - \xi(x)]} :\, \chi(x) \;=\; \Psi(x)\, \omega(x) \,, \tag{3.24}$$

where

$$\Psi(x) \;=: e^{\displaystyle i(1+\gamma^5)\, \sqrt{\tfrac{\pi}{a-1}}\, \Sigma(x)} \;:\, \chi(x) \,, \tag{3.25}$$

$$\omega(x) =: e^{\,i\,(1+\gamma^5)\,\sqrt{\frac{\pi}{a-1}}\,(\xi'' - \eta)} :, \tag{3.26}$$

$$\chi(x) = \Big(\frac{\mu_o}{2\pi}\Big)^{1/2} e^{-i\frac{\pi}{4}\gamma^5} : e^{i\,\sqrt{\pi}\{\,\gamma^5\,\varphi(x) + \int_{x^1}^{\infty}\,\dot{\varphi}(x^0, z^1)\,dz^1\}} : . \tag{3.27}$$

The gauge field can be written as

$$\mathcal{A}_\mu(x) \,=\, \Sigma_\mu(x) \,+\, \frac{1}{e}\,\omega^{-1}(x)\,\partial_\mu\,\omega(x)\,. \tag{3.28}$$

where

$$\Sigma_\mu(x) \,\doteq\, -\frac{1}{m_a}\Big\{\,\epsilon_{\mu\nu}\,\partial^\nu \,+\, \frac{1}{a-1}\,\partial_\mu\,\Big\}\,\Sigma(x)\,. \tag{3.29}$$

The vector current is given by

$$J_\mu = m_a\,\epsilon_{\mu\nu}\partial^\nu\,\Sigma \,+\, L^\mu\,, \tag{3.30}$$

where L_μ is a longitudinal current

$$L_\mu = -\frac{1}{\sqrt{\pi}}\Big\{\,(\partial_\mu \,+\, \tilde{\partial}_\mu)\varphi \,+\, \frac{a}{\sqrt{a-1}}\,\tilde{\partial}_\mu\eta \,-\, \frac{1}{\sqrt{a-1}}\,[\,(a-1)\,\partial_\mu \,-\, \tilde{\partial}_\mu]\xi''\Big\}\,. \tag{3.31}$$

For $M_o \,=\, 0$ we recover the operator solution of the anomalous model with massless fermions discussed in Refs. [5,12].

Due to the opposite metric quantization for the coupled sine-Gordon fields ξ'' and η, the massless free field combination $(\xi'' - \eta)$ generates zero norm states from the vacuum

$$\|(\xi''(x) - \eta(x))|\Omega\rangle\|^2 = 0\,. \tag{3.32}$$

As a consequence, the field $\omega(x)$ generates constant contributions to the Wightman functions

$$\langle\Omega|\prod_{i=1}^{n}\omega^*(x_i)\,\prod_{j=1}^{n}\omega(y_j)|\Omega\rangle = 1\,. \tag{3.33}$$

Due to the presence of the longitudinal current L_μ in Eq. (3.30), the Gauss' law holds in the weak form and is satisfied on the physical subspace $\mathcal{H}^{phys}$, which is defined by the subsidiary condition

$$\langle\Phi|(\partial_\mu \mathcal{F}^{\mu\nu}(x) + J^\nu(x))|\Psi\rangle = \langle\Phi|L^\nu(x)|\Psi\rangle \ , \ \forall|\Phi\rangle, |\Psi\rangle \in \mathcal{H}^{phys} . \tag{3.34}$$

In a genuine gauge theory, the algebra of physical operators $\Im^{phys}$ must be identified with the subalgebra of $\Im$ which obeys the subsidiary condition and the physical Hilbert space is defined by gauge invariant states accomodated as equivalent classes modulo $L_\mu(x)$. However, it is a peculiarity of two-dimensional anomalous chiral models [5,6] that the algebra $\Im^{phys} \equiv \Im$, since all operators belonging to the intrinsic field algebra $\Im$ commute with the longitudinal current,

$$[\mathcal{O}, L_\mu] = 0 \ , \ \forall \mathcal{O} \in \Im . \tag{3.35}$$

The chiral anomaly promotes the intrinsic set of fields $\{\psi, \psi, \mathcal{A}_\mu\}$ to the status of physical operators, which must be singlet under chiral operator gauge transformations. We shall return to this point further on.

It is very instuctive to compare the mass term of the anomalous model, appearing in (3.20) with that of the gauge model (2.53) with $g = 0$. For the Schwinger model we have $\beta = 2\sqrt{\pi}$, $\Phi \equiv \varphi$, $\eta_a \equiv \eta = \phi''$, and the mass operator is given by

$$: \cos \Big\{ 2\sqrt{\pi}\,\Sigma + 2\sqrt{\pi}(\varphi - \eta) \Big\} : . \tag{3.36}$$

The spurious field combination

$$\hat{\sigma} =: e^{2i\sqrt{\pi}(\varphi-\eta)} : ,$$

carries the free fermion selection rule and the violation of the asymptotic factorization property of the Wightman functions leads to the θ-vacuum structure [20,21,23,24].

In the anomalous model, the spurious field combination $(\xi'' - \eta)$ carries no fermion selection rule, which leads to the absence of vacuum degeneracy. For $a = 2$ the mass operator of the anomalous model can be written as

$$: \cos \Big\{ 2\sqrt{\pi}\,\Sigma + 2\sqrt{\pi}\varphi + 2\sqrt{\pi}\,(\xi'' - \eta) \Big\} : . \tag{3.37}$$

The presence of the field ξ'' in the mass operator (3.37) explicitly shows that for $a = 2$ the anomalous chiral model with massive fermions cannot be considered as equivalent to the massive

QED_2, as has been claimed [10] for the corresponding models with massless fermions. For $M_o = 0$, the Bose fields ξ'' and η becomes free massless fields. Although in this case we can write the chiral density operator as

$$[\psi_\ell^* \chi_r] \, =: e^{\,2i\sqrt{\pi}\Sigma} : \hat{\sigma} \; : e^{\,2i\sqrt{\pi}\xi''} : , \tag{3.38}$$

the field ξ'' cannot be extracted from the field algebra and the field $\hat{\sigma}$ cannot be defined on the Hilbert space of the anomalous chiral model [5,6].

In the gauge model previously considered, the contributions of the field η, quantized with negative metric, cancels those arising from the field φ and which carries the fermion degrees of freedom. As is well known [20–23], this charge-screening mechanism leads to the vacuum degeneracy and the θ-vacuum parametrization. However, in the anomalous model, the appearance in the mass operator of the massless free field combination $(\xi'' - \eta)$, has no physical consequences on the vacuum structure of the model. This can be shown by considering for example the Wightman functions of the mass operator $[\bar{\psi}(x)\psi(x)]$. Using the functional integral Gell-Mann and Low formula, the functional integration over the field η cancels those arising from the integration over the field ξ'' and we get,

$$\langle \Omega | [\bar{\psi}(x_1)\psi(x_1)] \cdots [\bar{\psi}(x_n)\psi(x_n)] | \Omega \rangle =$$

$$\langle \Omega | : \cos \Big\{ 2\sqrt{\frac{\pi}{a-1}}\, \Sigma(x_1) \, + \, 2\sqrt{\pi}\varphi(x_1) \, + \, 2\sqrt{\frac{\pi}{a-1}}\, (\, \xi''(x_1) - \eta(x_1) \,) \Big\} : \cdots$$

$$: \cos \Big\{ 2\sqrt{\frac{\pi}{a-1}}\, \Sigma(x_n) \, + \, 2\sqrt{\pi}\varphi(x_n) \, + \, 2\sqrt{\frac{\pi}{a-1}}\, (\, \xi''(x_n) - \eta(x_n) \,) \Big\} : | \Omega \rangle \equiv$$

$$\langle \Omega | : \cos \Big\{ 2\sqrt{\frac{\pi}{a-1}}\, \Sigma(x_1) \, + \, 2\sqrt{\pi}\varphi(x_1) \Big\} : \cdots : \cos \Big\{ 2\sqrt{\frac{\pi}{a-1}}\, \Sigma(x_n) \, + \, 2\sqrt{\pi}\varphi(x_n) \Big\} : | \Omega \rangle =$$

$$= \langle \Omega | [\overline{\Psi}(x_1)\Psi(x_1)] \cdots [\overline{\Psi}(x_n)\Psi(x_n)] | \Omega \rangle \, . \tag{3.39}$$

For the general Wightman functions obtained with polynomials of the fundamental fields defining the model, we obtain the isomorphism

$$\langle\Omega|\mathcal{P}(\bar{\psi},\psi,\mathcal{A}_\mu)|\Omega\rangle \equiv \langle\Omega|\mathcal{P}(\overline{\Psi},\Psi,\Sigma_\mu)|\Omega\rangle\,, \tag{3.40}$$

implying that the quantum dynamics and the net physical content of the model are carried by the fields $\{\bar{\Psi},\Psi,\Sigma_\mu\}$.

C. Wess-Zumino field and extended gauge invariance

Although the Bose fields ξ', η and ω gives no contributions to the general Wightman functions of the anomalous chiral model, they play an important role in the field algebra and in the structure of the Hilbert space. In this section we shall display the role played by these fields and the Wess-Zumino (W-Z) field in order to ensure the invariance of the field algebra under extended local gauge transformations.

To begin with, consider the action of the anomalous model in terms of the field variables $\{U,V\}$,

$$\begin{aligned} S[U,V,\bar{\chi},\chi] &= S[UV] + \Big(\frac{a}{2}-1\Big)\Gamma[V] + \frac{a}{2}\Gamma[U] + \\ &+\bar{\chi}\, i\,\partial\!\!\!/\,\chi - M_o\int d^2z\Big(\chi_r^*\chi_\ell V^{-1} + \chi_\ell^*\chi_r V\Big)\,. \end{aligned} \tag{3.41}$$

The $U(1)$ group-valued Bose fields $\{U,V\}$ can be factorized as,

$$U = \bar{U}\,h\ \ ,\ \ V = h^{-1}\bar{V}\,, \tag{3.42}$$

where $\{\bar{U},\bar{V}\}$ depends on the field $\phi' = \Sigma - \eta$,

$$\bar{U} = e^{-2i\,\frac{(a-2)}{a}\sqrt{\frac{\pi}{a-1}}\,\phi'}\,, \tag{3.43}$$

$$\bar{V} = e^{-2i\sqrt{\frac{\pi}{a-1}}\,\phi'}\,, \tag{3.44}$$

and the field variable h carries the dependence on the field ξ'',

$$h = e^{2i\sqrt{\frac{\pi}{a-1}}\,\xi''}\,. \tag{3.45}$$

The action (3.41) can be written in terms of the fields $\{\bar{U}, \bar{V}, h\}$. Using the P-W identity (2.22), and the fact that

$$(a-2)\bar{V}\partial_\mu \bar{V}^{-1} - a\bar{U}\partial_\mu \bar{U}^{-1} = 0\,, \tag{3.46}$$

we obtain,

$$S[U,V,\bar{\chi},\chi] \equiv S[\bar{U},\bar{V},h,\bar{\chi},\chi] = S[\bar{U}\bar{V}] + (a-1)\Gamma[h] + \Big(\frac{a}{2}-1\Big)\Gamma[\bar{V}] + \frac{a}{2}\Gamma[\bar{U}] +$$

$$+\bar{\chi}\, i\, \partial\!\!\!/\, \chi - M_o \int d^2z \Big(\chi_r^* \chi_\ell \bar{V}^{-1} h + \chi_\ell^* \chi_r h^{-1} \bar{V}\Big)\,. \tag{3.47}$$

The "embedded" version of the model is obtained by the introduction of the W-Z field by performing the chiral gauge transformations [7]

$${}^g\psi_\ell(x) = g(x)\psi_\ell(x)\,, \tag{3.48}$$

$${}^g\mathcal{A}_\mu(x) = \mathcal{A}_\mu(x) + \frac{1}{e}\, g(x)\, i\, \partial_\mu\, g^{-1}(x)\,, \tag{3.49}$$

where $g(x)$ is given in terms of the W-Z field $\theta(x)$,

$$g(x) = e^{-i\,(1+\gamma^5)\,\sqrt{\pi}\,\theta(x)}\,. \tag{3.50}$$

The gauge transformation acts on the Bose fields $\{U, V\}$ as,

$${}^gU = Ug \quad , \quad {}^gV = g^{-1}V \quad , \quad {}^gG = G\,. \tag{3.51}$$

The fields $\{\,{}^g\bar{\psi}_\ell,\, {}^g\psi_\ell,\, {}^g\mathcal{A}_\mu\}$ are manifest invariant under extended gauge transformations:

$$U \rightarrow U\tilde{g}\ ,\ V \rightarrow \tilde{g}^{-1}V\ ,\ g \rightarrow g\tilde{g}^{-1}\,. \tag{3.52}$$

Rescalling the W-Z field,

$$\theta' = (a-1)^{1/2}\theta\ \ ,\ \ g = e^{-\,2i\sqrt{\frac{\pi}{a-1}}\,\theta'}\,, \tag{3.53}$$

the gauge transformed variables $\{\,{}^gU,\,{}^gV\}$ are given by

$$ {}^gU = Ug = \bar{U}hg\,, \tag{3.54} $$

$$ {}^gV = g^{-1}V = (hg)^{-1}\bar{V}\,. \tag{3.55} $$

Introducing the field

$$ \xi''' \doteq \xi'' - \theta'\,, \tag{3.56} $$

we can define a new field variable $\tilde{h}$,

$$ \tilde{h} = e^{2i\sqrt{\frac{\pi}{a-1}}\,(\xi''-\theta')} = e^{2i\sqrt{\frac{\pi}{a-1}}\,\xi'''}\,. \tag{3.57} $$

In this way, we obtain the algebraic isomorphism

$$ {}^gU = \bar{U}\tilde{h} \sim \bar{U}h = U\,, \tag{3.58} $$

$$ {}^gV = \tilde{h}^{-1}\bar{V} \sim h^{-1}\bar{V} = V\,, \tag{3.59} $$

showing that, from the algebraic point of view the two descriptions are indeed equivalent.

For the general Wightman functions of the field operators belonging to the intrisic field algebra we obtain

$$ \langle\Omega|\,{}^g\bar{\psi}(x_1)\cdots{}^g\bar{\psi}(x_n){}^g\psi(y_1)\cdots{}^g\psi(y_n)|\Omega\rangle \equiv \langle\Omega|\bar{\psi}(x_1)\cdots\bar{\psi}(x_n)\psi(y_1)\cdots\psi(y_n)|\Omega\rangle\,, \tag{3.60} $$

$$ \langle\Omega|{}^g\mathcal{A}_\mu(x_1)\cdots{}^g\mathcal{A}_\mu(x_n)|\Omega\rangle \equiv \langle\Omega|\mathcal{A}_\mu(x_1)\cdots\mathcal{A}_\mu(x_n)|\Omega\rangle\,, \tag{3.61} $$

expressing the equivalence between the gauge transformed intrinsic set of fields $\{\,{}^g\bar{\psi},\,{}^g\psi,\,{}^g\mathcal{A}_\mu\}$ and $\{\bar{\psi},\psi,\mathcal{A}_\mu\}$:

$$ {}^g\psi(x) \;=\; \Psi(x)\,\tilde{\omega}(x) \;\sim\; \psi(x)\,, \tag{3.62} $$

$$ {}^g\mathcal{A}_\mu(x) \;=\; \Sigma_\mu(x) \;+\; \frac{1}{e}\,\tilde{\omega}^{-1}(x)\,\partial_\mu\,\tilde{\omega}(x) \;\sim\; \mathcal{A}_\mu(x)\,, \tag{3.63} $$

where the field $\tilde{\omega}(x)$ is written in terms of ξ''',

$$\tilde{\omega}(x) =: e^{i\,2\,\sqrt{\frac{\pi}{a-1}}\,(\xi''' - \eta)} : . \tag{3.64}$$

This implies the isomorphism between the field algebras

$${}^{g}\Im \sim \Im\,. \tag{3.65}$$

In this way, for any functional $\boldsymbol{\mathcal{F}}(\bar{\psi}, \psi, \mathcal{A}_\mu)$, we obtain

$$\langle\, \boldsymbol{\mathcal{F}}({}^{g}\!\mathcal{A}_\mu,\, {}^{g}\bar{\psi},\, {}^{g}\psi)\,\rangle \;=\; \langle\, \boldsymbol{\mathcal{F}}(\mathcal{A}_\mu, \bar{\psi}, \psi)\,\rangle\,, \tag{3.66}$$

expressing the isomorphism of the Hilbert spaces:

$${}^{g}\mathcal{H} \sim \mathcal{H}\,. \tag{3.67}$$

In order to display the role played by the W-Z field in the model with massive Fermi fields, consider the action corresponding to the model defined in terms of the gauge transformed fields $\{\, {}^{g}\bar{\psi}_\ell,\, {}^{g}\psi_\ell,\, {}^{g}\!\mathcal{A}_\mu\}$,

$$\begin{gathered} S[Ug, g^{-1}V, \bar{\chi}, \chi] = S[UV] + \Big(\frac{a}{2} - 1\Big)\Gamma[gV] + \frac{a}{2}\Gamma[Ug^{-1}] + \\ \bar{\chi}\, i\, \partial\!\!\!/\, \chi - M_o \int d^2 z \Big(\chi_r^* \chi_\ell V^{-1} g + \chi_\ell^* \chi_r g^{-1} V\Big)\,. \end{gathered} \tag{3.68}$$

Since we have performed an operator chiral gauge transformation, the mass term is not manifest invariant and the W-Z field obeys a coupled sine-Gordon equation. Using the P-W identity, we can write the gauge transformed action (3.68) as

$$\begin{gathered} S[Ug, g^{-1}V, \bar{\chi}, \chi] = S_{WZ}[U, V, g] + S[UV] + \Big(\frac{a}{2} - 1\Big)\Gamma[V] + \frac{a}{2}\Gamma[U] + \\ + \bar{\chi}\, i\, \partial\!\!\!/\, \chi - M_o \int d^2 z \Big(\chi_r^* \chi_\ell V^{-1} g + \chi_\ell^* \chi_r g^{-1} V\Big)\,, \end{gathered} \tag{3.69}$$

where $S_{WZ}[U, V, g]$ is the W-Z action for the anomalous model with massless Fermi fields ($M_o = 0$) and is given by

$$S_{WZ}[U,V,g] = (a-1)\Gamma[g] + \frac{1}{4\pi}\int d^2z\Big\{\Big(\frac{a}{2}-1\Big)V\partial^\mu V^{-1} - \frac{a}{2}U\partial^\mu U^{-1}\Big\}\,g\partial_\mu g^{-1}\,. \qquad (3.70)$$

In terms of the factorized fields $\{\bar U, \bar V, h\}$, the WZ action can be written as

$$S_{WZ}[U,V,g] \equiv S_{WZ}[g,h] = (a-1)\Gamma[hg] - (a-1)\Gamma[g] - (a-1)\Gamma[h]\,. \qquad (3.71)$$

From (3.69) and (3.71) we obtain the gauge-transformed action as

$$S[gU, g^{-1}V, \bar\chi, \chi] = S[\bar U, \bar V, \tilde h, \bar\chi, \chi] =$$
$$= S[\bar U\bar V] + (a-1)\Gamma[\tilde h] + \Big(\frac{a}{2}-1\Big)\Gamma[\bar V] + \frac{a}{2}\Gamma[\bar U] + \bar\chi\, i\,\partial\!\!\!/\,\chi - M_o\int d^2z\Big(\chi_r^*\chi_\ell \bar V^{-1}\tilde h + \chi_\ell^*\chi_r \tilde h^{-1}\bar V\Big)\,, \qquad (3.72)$$

which is identical to the action (3.47) with the field ξ'' being replaced by the field ξ''', and thus

$$S[Ug, g^{-1}V, \bar\chi, \chi] \equiv S[U, V, \bar\chi, \chi]\,. \qquad (3.73)$$

¿From the functional integral point of view, the isomorphism of the Hilbert spaces ${}^g\mathcal{H}$ and $\mathcal{H}$, follows from the equivalence between the corresponding generating functionals. As a consequence of the algebraic isomorphism ${}^g\Im \sim \Im$, according to (3.62) and (3.63), the source terms associated with the intrinsic set of fields are singlets under extended gauge transformations. Using (3.58), (3.59),(3.73), we obtain

$$\langle e^{i\,[\,{}^g\bar\psi\,\upsilon \,+\, \bar\upsilon\,{}^g\psi \,+\, {}^g\!\mathcal{A}_\mu\,\rho^\mu\,]}\rangle_\Omega = \langle e^{i\,[\,\bar\Psi\,\tilde\omega^*\,\upsilon \,+\, \bar\upsilon\,\Psi\,\tilde\omega \,+\, (\Sigma_\mu + \frac{1}{e}\tilde\omega^{-1}\partial_\mu\tilde\omega)\rho^\mu\,]}\rangle_\Omega \equiv$$
$$\langle e^{i\,[\,\bar\Psi\,\omega^*\,\upsilon \,+\, \bar\upsilon\,\Psi\,\omega \,+\, (\Sigma_\mu + \frac{1}{e}\omega^{-1}\partial_\mu\omega)\rho^\mu\,]}\rangle_\Omega \equiv \langle e^{i\,[\,\bar\Psi\,\upsilon \,+\, \bar\upsilon\,\Psi \,+\, \Sigma_\mu\,\rho^\mu\,]}\rangle_\Omega\,, \qquad (3.74)$$

which implies the equivalence between the generating functionals

$$\langle e^{i\,[\,{}^g\bar\psi\,\upsilon \,+\, \bar\upsilon\,{}^g\psi \,+\, {}^g\!\mathcal{A}_\mu\,\rho^\mu\,]}\rangle_\Omega = \langle e^{i\,[\,\bar\psi\,\upsilon \,+\, \bar\upsilon\,\psi \,+\, \mathcal{A}_\mu\,\rho^\mu\,]}\rangle_\Omega\,. \qquad (3.75)$$

We conclude that, even in the anomalous chiral model with massive Fermi fields, the generating functional is invariant under extended gauge transformations. This implies the isomorphism between the Hilbert space $\mathcal{H}$ of the anomalous model, defined in terms of the intrinsic fields $\{\bar{\psi}, \psi, \mathcal{A}_\mu\}$ and which are not manifest gauge invariant (gauge non-invariant formulation), and the Hilbert space ${}^g\mathcal{H}$ construct in terms of the fields $\{\,{}^g\bar{\psi}_\ell,\, {}^g\psi_\ell,\, {}^g\!\mathcal{A}_\mu\}$ and which are manifest invariant under extended gauge transformations (gauge invariant formulation). The role played by the W-Z field in the anomalous chiral model with massive fermions is exactly the same played by it in the corresponding model with massless fermions. The introduction of the W-Z field in the anomalous chiral model replicates the theory, changing neither its algebraic structure nor its physical content. This streamlines the conclusions of Refs. [5,12] for the anomalous model with massless Fermi fields.

IV. CONCLUSIONS

Using a synthesis of the functional integral and operator formulations, we have considered some structural aspects of the fermion-boson mapping in two-dimensional gauge and anomalous gauge models. We have analyzed the role played by the "decoupled" free massless Bose fields, which appear in bosonized models with massless fermions, in the corresponding models with massive Fermi fields.

For the QED_2 with current-current interaction among massive fermions, the use of an auxiliary vector field to reduce the Thirring interaction, introduces a redundant Bose field algebra which is insensitive to the presence of the mass term for the Fermi fields. This procedure leads to the appearance of the longitudinal current ℓ_μ, which generates zero norm states from the vacuum.

For non Abelian models in $2+1$ dimensions, the Wilson loop operator [1,2] defined by,

$$\exp\{g \int_\Gamma \bar{\psi}\gamma^\mu\psi dx_\mu\}\,,$$

plays an important role in order to establish the fermion-boson correspondences. In two-dimensional Abelian models, due to the trivial topology concerned, in the computation of the expectation value of the loop operator,

$$\langle \exp\{g \int_\Gamma \bar\psi\gamma^\mu\psi dx_\mu\}\rangle = \langle \exp\{\int_\Gamma (B^\mu - \ell^\mu) dx_\mu\}\rangle \,,$$

the contributions of the current ℓ_μ reduces to the identity. However, for non Abelian models, it seems to be very intructive to make a foundational investigation of the structural properties of the fermion-boson mappings in $2+1$ dimensions, which may offer a valuable lesson for the understanding of the underlying physical properties of the higher dimensional field theory models. An interesting mathematical and structural question, that must be presumably relevant for the extension of the bosonization procedure to non-Abelian models in $2+1$ dimensions [1], where the knot invariants with non trivial topology are present, is related to whether these loops operators belongs to the intrinsic field algebra and can be defined in the Hilbert space of the theory. Another question, that until recent publications has been not fully clarified, concerns to the appearance of a local gauge symmetry in the bosonized version of the Thirring model in $2+1$ dimensions. A clear understanding of these points seems to us essential in order to ensure that the fermion-boson mapping is established on the Hilbert space of states and thus may offer information about the true physical content of the original theory.

In the anomalous chiral QED_2 with massive Fermi fields we show that the original decoupled massless Bose fields quantized with opposite metric of the chiral model with massless Fermi fields, are promoted to fields with non trivial dynamics governed by the sine-Gordon equation. Nevertheless, their combination remains a free massless field with zero norm, which contributes with a phase to the mass operator. Contrary to what happens in the genuine gauge model, this phase factor carries no fermion selection rule and no vacuum degeneracy is implied by the appearance of this spurious field with zero scale dimension. We have used the introduction of the mass term for the Fermi fields as an alternative and practical way for probing our previous conclusions of Refs. [5,6]. In that case, since the Fermi field is massless, the decoupled massless Bose field appears at a first glance to play merely a spectator role. However, the apparently decoupled Bose field plays an important role in the construction of the Hilbert space of the anomalous chiral model. The naive extraction of this apparently decoupled massless free field from the field algebra, by factorizing the bosonized partition function, leads to misleading conclusions concerning to the physical content of the model, such as the violation of the asymptotic factorization property of the Wightman functions, the need of the θ-vacuum parametrization and the equivalence of the chiral model defined for $a = 2$ and the vector model, as proposed in Ref. [10]. Even in the anomalous model with massive Fermi fields, the introduction of the W-Z field only replicates the field algebra of the theory, changing neither its algebraic structure not its physical content. As expected from the analysis of the massless model [5,6], the Hilbert space of the anomalous chiral model with massive fermions and defined for $a = 2$ does not contains as a proper subspace the Hilbert space of the corresponding gauge model.

The authors are gratefull to CNPq-Brazil, FAPESP-Brazil and CAPES-Brazil for partial financial support. One of us (LVB) is gratefull to R. L. P. G. Amaral and K. D. Rothe for many stimulating discussions.

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